WATER CLOCKS- EXPERIMENTAL VERIFICATION OF TORRICELLI'S LAW

Aim

The main aim of the experiment is to

- Study the basics of Design of Experiments and ANOVA table.
- Validate Torricelli's law.

Apparatus Required

Bottom closed Graduated Glass Pipe, Stopwatch, and Beaker.

Representation

Each system considered in this experiment is a cylindrical pipe with the top end open and the bottom end closed. Holes were created on the side of the pipe through which water exits the system.

The following definitions will be used throughout the experiment shown in figure 1.

I. Parameters:

- R, the inner radius of the pipe
- r, the radius of all draining holes for that particular pipe
- h, the height of the column of water (from the bottom of the lowest hole)
- V, the volume of the column of water
- v, the velocity of water as it escapes through the holes
- f, the distance between the bottom of the hole and the bottom of the tube
- A, the cross-sectional area of the pipe
- a, the cross-sectional area of each hole in the pipe

II. Functions:

- $h(t)$: height of the water column
- $V(t)$: volume of water column
- $v(t)$: exit velocity

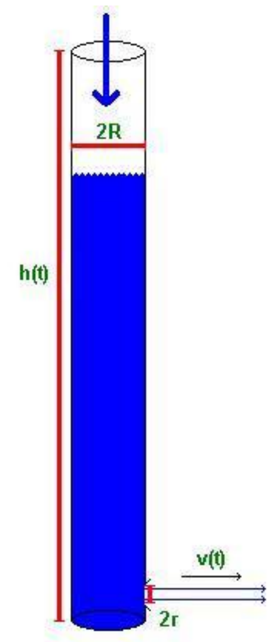

Figure 1. Toricelli law parameters

Theory

Consider $\frac{dV}{dt}$, the rate at which the volume of the water column changes. One way we can describe this is by considering the amount of water leaving the system in that length of time, *av*. We thus have that $\frac{dV}{dt} = -av$. We can also describe this by using the Chain Rule, $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$. Since for this system $dV = Adh$, we have that $\frac{dV}{dt} = A\frac{dh}{dt}$. We now set the two expressions for $\frac{dV}{dt}$ equal to each other:

$$\frac{dV}{dt} = -av = A\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{-av}{A}$$

Since the cross-sections of both the pipe and the hole are circular, we have that $a = \pi r^2$ and $A = \pi R^2$, and the problem simplifies to

$$\frac{dh}{dt} = \frac{-\pi r^2 v}{\pi R^2} = \frac{-r^2 v}{R^2}.$$

We do not know $v(t)$, but we do know from Torricelli's Law that $v(h) = \sqrt{2gh}$ for an ideal draining system, where g is the gravitational constant. However, our system is not ideal. For example, viscosity of the liquid must be considered. Any rotation inside the pipe results in an energy loss, as well. Thus, a more general form of the law is appropriate. We thus use $v(h) = \alpha\sqrt{gh}$.

We now use this function for v to solve for $h(t)$. Note that we use the initial condition

$h(0) = h_0$,

the initial height of the water column:

$$\frac{dh}{dt} = \frac{-r^2 v}{R^2} = \frac{-r^2 \alpha \sqrt{gh}}{R^2}$$

$$\frac{1}{\sqrt{h}} \frac{dh}{dt} = \frac{-r^2 \alpha \sqrt{g}}{R^2}$$

$$2\sqrt{h} = \frac{-r^2 \alpha \sqrt{g}}{R^2} t + C$$

$$h(0) = h_0 :$$

$$2\sqrt{h_0} = C$$

$$2\sqrt{h} = \frac{-r^2 \alpha \sqrt{g}}{R^2} t + 2\sqrt{h_0}$$

$$\sqrt{h} = \frac{-r^2 \alpha \sqrt{g}}{2R^2} t + \sqrt{h_0}$$

$$h(t) = \left(\frac{-r^2 \alpha \sqrt{g}}{2R^2} t + \sqrt{h_0} \right)^2 = \frac{r^4 \alpha^2 g}{4R^4} t^2 - \frac{r^2 \alpha \sqrt{gh_0}}{R^2} t + h_0$$

A point of particular significance is the point where *h*(*t*) = 0. Beyond that point, the function stays at zero. We can thus make a piecewise graph to model the function, but first we need to find the value of *t* at which the water column has fully emptied the pipe:

$$h(t) = \left(\frac{r^4 \alpha^2 g}{4R^4} \right) t^2 + \left(-\frac{r^2 \alpha \sqrt{gh_0}}{R^2} \right) t + h_0 = 0$$

$$t = \frac{\frac{r^2 \alpha \sqrt{gh_0}}{R^2} \pm \sqrt{\left(-\frac{r^2 \alpha \sqrt{gh_0}}{R^2} \right)^2 - 4\left(\frac{r^4 \alpha^2 g}{4R^4} \right) h_0}}{2\left(\frac{r^4 \alpha^2 g}{4R^4} \right)} = \frac{\frac{r^2 \alpha \sqrt{gh_0}}{R^2} \pm \sqrt{\left(\frac{r^4 \alpha^2 gh_0}{R^4} \right) - \left(\frac{r^4 \alpha^2 gh_0}{R^4} \right)}}{\frac{r^4 \alpha^2 g}{2R^4}}$$

$$t = \frac{\frac{r^2 \alpha \sqrt{gh_0}}{R^2}}{\frac{r^4 \alpha^2 g}{2R^4}} = \frac{2r^2 R^4 \alpha \sqrt{gh_0}}{r^4 R^2 \alpha^2 g} = \frac{2R^2}{r^2 \alpha} \sqrt{\frac{h_0}{g}}$$

In fact, the point where *h*(*t*) = 0 is the vertex of the parabola, as it is at that point where *dh*/*dt* = 0.

We also have information to solve for $v(t)$ and $V(t)$:

$$v(t) = \alpha\sqrt{gh(t)} = \alpha\sqrt{g\left(\frac{-r^2\alpha\sqrt{g}}{2R^2}t + \sqrt{h_0}\right)^2} = \alpha\sqrt{g}\left(\frac{-r^2\alpha\sqrt{g}}{2R^2}t + \sqrt{h_0}\right) = \frac{-r^2\alpha^2 g}{2R^2}t + a\sqrt{gh_0}$$

$$V(t) = \pi R^2 h(t) = \pi R^2\left[\left(\frac{r^4\alpha^2 g}{4R^4}\right)t^2 + \left(-\frac{r^2\alpha\sqrt{gh_0}}{R^2}\right)t + h_0\right] = \left(\frac{\pi r^4\alpha^2 g}{4R^2}\right)t^2 + \left(-\pi r^2\alpha\sqrt{gh_0}\right)t + \pi R^2 h_0$$

We now know everything about the behavior of this system. Given r, R, and h_0, we know the following:

$$h(t) = \begin{cases} \left(\frac{r^4\alpha^2 g}{4R^4}\right)t^2 + \left(-\frac{r^2\alpha\sqrt{gh_0}}{R^2}\right)t + h_0, t \le t_f \\ 0, t > t_f \end{cases}$$

$$v(t) = \begin{cases} \frac{-r^2\alpha^2 g}{2R^2}t + \alpha\sqrt{gh_0}, t \le t_f \\ 0, t > t_f \end{cases}$$

$$V(t) = \begin{cases} \left(\frac{\pi r^4\alpha^2 g}{4R^2}\right)t^2 + \left(-\pi r^2\alpha\sqrt{gh_0}\right) t + \pi R^2 h_0, t \le t_f \\ 0, t > t_f \end{cases}$$

We note that $h(t)$ and $V(t)$ are quadratic, while $v(t)$ is linear.

PROCEDURE

The pipe used in this system has one hole near the bottom of the pipe through which water exits the system (Figure 1). Four different pipes were used, each with a different combination of values for r and R. With the hole plugged, water is poured into the pipe up to the top. The hole is then unplugged, such that water can exit through the hole. We hope to model the resulting behavior of the system and indirectly verify Torricelli’s Law.

Experiment

For all experiments involving this system, we use one pipe. For the pipe, the hole was plugged as the pipe was filled with water to the top. Time was kept beginning from the moment the hole was unplugged; making sure that the pipe was always perpendicular to the ground. The time *t* at regular intervals of height till the water is completely drained is recorded in table 1. Three trials were taken for each pipe. All data are listed in the table below. Theoretical values were computed through the formula

We note that the percent error for each pipe is very low, less than 20% in all cases and less than 10% in three cases, so this provides evidence to support Torricelli's Law, as the radii were varied with each pipe. We notice that the percent error generally increases with the time it took to drain the pipe, which may mean that the value given for α may be too low. We rework the formula above to calculate a more accurate value of α from the experimental values. Using the graphs drawn regression equation is found. From that equation we find the actual value of α using back substitution method.

Sl.No.	Height of the pipe in m	Time(Seconds) Pipe Diameter =___ m			
		Trail 1	Trial 2	Trial 3	Mean
1					
2					
3					
4					
5					
6					
7					

Table 1.Measured values of time for corresponding pipes

Results and calculations:

From the formulae below calculate the respective theoretical values of height, velocity and volume required using the time in seconds for various diameters of pipes. The calculated values are inserted in table 2.

Theoretical values

$$\sqrt{h(t)} = \begin{cases} \left(\frac{-r^2 \alpha \sqrt{g}}{2R^2} \right) t + \sqrt{h_0}, t \le t_f \\ 0, t > t_f \end{cases}$$

$$v(t)_{(Ideal)} = \sqrt{2gh}$$

$$V(t)_{(Expected)} = \begin{cases} \left(\frac{\pi r^4 \alpha^2 g}{4R^2} \right) t^2 + \left(-\pi r^2 \alpha \sqrt{g h_0} \right) t + \pi R^2 h_0, t \le t_f \\ 0, t > t_f \end{cases}$$

The experimental values are calculated as below

Experimental velocity $v = \alpha \sqrt{(gh)}$ (m/s)

Experimental volume $V = \pi R^2 h$ (m^3)

Sl.No.	Height of the pipe (m)	Time(Seconds) Pipe Diameter =__ mm				Velocity of water (m/s)		Volume of water (m^3)		Height (m)
		Trail 1	Trial 2	Trial 3	Mean	Experimental	Ideal	Experimental	Expected	Experimental
1										
2										
3										
4										
5										
6										
7										

Table 2. Calculation

Graphs :

Using MS EXCEL® ,

1)Plot the graph between $\sqrt{h}$ Vs. time and find the value of α .

2)Plot the graph between ideal and experimental velocity with time.

3) Plot the graph between experimental and expected volume with time.

<u>To find α:</u>

Find the value of α from the plot $\sqrt{h}$ Vs. time and volume Vs. time.

Data Analysis:

The data is analysed using design of experiments.

H_0: There is no significant difference in the velocity between the diameters of pipes

H_1: There is significant difference in the velocity between the diameters of pipes

Result:

From this experiment the Toricelli law governing the velocity of liquids through pipes are verified.