Buffon’s Needle Problem

Aim :

1. To find the approximate value of π experimentally.
2. To get exposed to the concepts of probability, trigonometry and calculus.

Equipments needed:

1. Paper
2. Needles – 100 each of two different sizes

Buffon’s Needle Problem:

A French nobleman, Le Comte de Buffon posed the problem in 1777. It is one of the oldest problems in geometric probability. The problem is, if a short needle of length l, is dropped on a paper that is ruled with equally spaced lines of distance $d \geq l$, then the probability that the needle comes to lie in a position where it crosses one of the lines is exactly $\frac{2l}{\pi d}$. One can get the approximate values of π using the above problem. That is, if a needle is dropped N times, and if the number of needles crossing at least one line is P, then $\frac{P}{N}$ should be approximately $\frac{2l}{\pi d}$. Hence, π should be approximated by $\frac{2lN}{dP}$. The above problem can be solved by evaluating an integral.

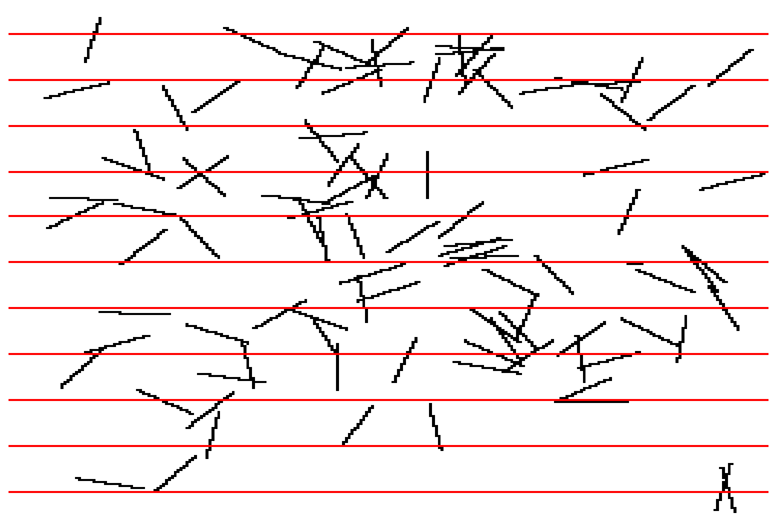

Figure 1. Buffon’s needle experiment

Let us say the needle drops to lie with an angle of θ away from the horizontal, where $0 \leq \theta \leq \pi/2$. A needle that lies with angle θ has height $l \sin \theta$. The probability that such a needle crosses one of the horizontal lines of distance d is $\frac{l \sin \theta}{d}$. Averaging over the possible angles θ, we get

$$p = \frac{2}{\pi}\int_0^{\pi/2} \frac{l\sin\theta}{d}\,d\theta$$

For a long needle, as long as $l\sin\theta \le d$, we obtain the same probability $\frac{l\sin\theta}{d}$. That is $0 \le \theta \le \sin^{-1}\frac{d}{l}$ and probability 1 if $\theta > \sin^{-1}\frac{d}{l}$. Hence,

$$p = \frac{2}{\pi}\left(\int_0^{\sin^{-1}\frac{d}{l}} \frac{l\sin\theta}{d}\,d\theta + \int_{\sin^{-1}\frac{d}{l}}^{\pi/2} 1\,d\theta \right)$$

$$= 1 + \frac{2}{\pi}\left(\frac{l}{d}\left(1 - \sqrt{1 - \frac{d^2}{l^2}}\right) - \sin^{-1}\frac{d}{l}\right) \text{ for } l > d.$$

Procedure and Calculation:

1. Draw parallel lines in a paper which are d units apart
2. Take a needle of length l such that $l < d$ and drop it on to the paper
3. Count the number of needles (say k) which crosses a line.

Tables:

Number of crossing	Number of tosses
k_1	n
k_2	n
k_3	n

Table 1

S. No.	**Properties Needed For Experiment**	**Needle-1**	**Needle-2**
1	No. of Needles (n)	n1=	n2=
2	No. of Crossings (k)	k1=	k2=
3	Length of Needle (l)	l1=	l2=
4	Distance Between The Lines (d)	d1=	d2=

Table 2

Let the number of tosses of the needle be n. Hence the probability,

$$P(\textit{the needle crossing a line}) = \frac{\textit{no. of crossings}}{\textit{total no. of tosses}} = \frac{k_i}{n}, i = 1,2,3$$

Also, $P(\textit{the needle crossing a line}) = \frac{2l_i}{\pi d}$

Hence, $\pi \approx \frac{2nl_i}{k_i d}$

Percent Error = (|experimental value – accepted value|/accepted value)* 100

Questions:

1. Which needle gives a better approximation?
2. Derive the probability of a needle crossing a line for $l \leq d$.
3. Derive the probability of a needle crossing a line for $l > d$.

The experiment can also be done by replacing the parallel lines by concentric circles, grids of squares, etc. It is important to note that this problem has wide application in cancer cells, cell growth and differentiation, military strategies, chromosome positioning etc.